SECTION 1

Mathematics teaching and learning

WE LIVE IN an increasingly complex society that requires more and more sophisticated uses of well understood mathematical processes for individuals to make the most of the opportunities that this society can provide them. While an ability to recognise numbers, to add, subtract, multiply or divide fluently with them, and to use them in measurement, statistics or probability underpins the mathematics used in everyday life or work, it is no longer sufficient in itself as the machines we work, communicate and play with can readily carry out these tasks. A capacity to assess the sense and importance of results that are made and presented succinctly is at least as critical. As more advanced capacities are required, an emphasis in teaching has shifted from simply learning skills to understanding the thinking on which a range of processes are built so that they might be adapted to new and novel applications.

As the mathematics needed by most people has broadened from the straightforward calculations and measurement of the past, the term 'numeracy' has emerged to provide a more satisfactory description of the extended mathematical processes and understanding that are now required in everyday situations. At the same time, technological change, societal change and climate change, and the impact of these changes on individuals, are making the world a place where there are fewer certainties. These changes demand people who can make sense of the mathematics they use, the problem situations they meet, and the results they determine, apply and interpret. This requires mathematically literate adults who can analyse the pronouncements of scientific, political and business leaders when they use arguments and data to promote their own ends, and who can communicate their concerns in persuasive ways using quantitative data and reasoning. Hence the inclusion of statistics and probability as a separate and full strand of mathematics at all levels of schooling as a means of dealing with uncertainty and complexity, rather than simply leaving it as a branch of measurement out of which it has grown. Within measurement, children still need to understand the underpinning attributes of length and perimeter, area, volume and capacity, mass and weight, angle, temperature and time. Only with a full conceptualisation of measurement will an ability to measure accurately, use formulas, and make estimates and accurate calculations in everyday situations develop. Geometric knowledge, processes and insights, particularly a well developed spatial sense, are also crucial in everyday life and most occupations. Without adequate knowledge of 2 dimensional and 3 dimensional shapes, their presentation in diagrammatic form, and the relationships between these shapes, their representations and properties, it is not possible even to begin to solve many of the problems that will be faced in further mathematics, employment and real life situations.

If such a broad understanding and use of mathematical ideas is to be gained and extended across all years of schooling, it is essential that the various topics that have often been taught in isolation—number, algebra, geometry, measurement, statistics, probability and logic—are related to one another and that these links are both brought out in the initial teaching and called upon in using mathematics. As Ritchhart (1994, p. 19) insightfully put it, 'too often they are taught in a discrete and separate way when teachers attempt to cover the curriculum rather than uncover it'. Mathematics needs to be viewed more as a way of thinking than as a subject in which the products of someone's thoughts are learned and used without reference to the problems that gave rise to them or the forms in which they evolved. In order to promote numeracy, it is necessary to engage students in authentic mathematical tasks, games and investigations that require thinking and understanding rather than the memorisation of facts, procedures and techniques.

Understanding has always been a critical outcome of mathematical education, but it has come to be seen as much more crucial when an ability to solve problems is at the centre of mathematics learning. It is understanding that allows ideas and techniques to be adapted to new ends. A learned procedure can only be applied in the manner in which it was practised. The problems met in novel situations require that existing knowledge be transformed to meet new demands, or that the problem as stated be transformed to match the mathematical processes that have been developed previously.

At the same time, understanding is invaluable in allowing ideas to be developed more efficiently and effectively. For example, multiplication facts used to include 'elevens' and 'twelves' facts acquired through the repetition of 'tables'. Yet an ability to think in tens, fundamental to place value, shows that twelve is simply 'ten and two', so multiplying by twelve is a matter of combining tens and ones:

8 twelves is 8 tens and 8 twos, or 80 and 16—ie, 96

9 twelves is 9 tens and 9 twos, or 90 and 18—ie, 108

As many teachers noted, students often confused these 'facts', which were readily available by using knowledge of the basic facts involving 0–9. Further, once this way of thinking was available, any multiplication with twelve was available in the same way:

17 twelves is 170 and 34—ie, 204

18 twelves is 180 and 36—ie, 216

24 twelves is 240 and 48—ie, 288

Similarly, rather than acquire some facts by simply replacing the digits (eg, 5 elevens are 55, 6 elevens are 66), thinking in terms of 'ten and one' extends beyond the 'tables' to multiplication with numbers of all sizes:

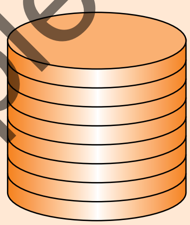
8 elevens is 80 and 8—ie, 88

14 elevens is 140 and 14—ie, 154 (not 1414)

23 elevens is 230 and 23—ie, 253 (not 2323)

Using earlier learning of addition and numeration allows learners to build up these new facts more efficiently. At the same time, they develop confidence in their ability and see number as a cohesive body of knowledge rather than as a series of fragmented ideas.

Similarly, when underlying geometrical ideas are developed in a meaningful way so that a high degree of spatial sense results, many aspects of measurement can be viewed as a bringing together of spatial knowledge with the numeration and computation understanding that readily allows measurement concepts to be quantified. For example, a cylinder may be viewed as a series of discs:



The volume of the cylinder can be given by summing the volume of the discs. Each disc has an area of πr^2, and its volume is given by multiplying this area by its thickness. But the height of all of the discs must be the same as the height of the cylinder (h). So the volume of the cylinder is given by multiplying the area of a disc by the total height of all of the discs; $\pi r^2 h$.

Other aspects of measurement call on number knowledge alone by using scales (temperature, mass, time) along with an understanding of the physical characteristics on which they are built. Of course, a good understanding of the metric system that is based on the place value and renaming ideas of Base 10 numeration is also critical. Thus, measurement as a whole can be derived from and connected to all the other aspects of early mathematics rather than conceived in isolation.

If all new ideas, processes and applications can be connected to existing knowledge in this way, not only will this mean the new concepts and strategies can be efficiently built out of established ways of thinking and proceeding, but the learning will also be more effective as an integrated view of mathematics is formed. Concepts and processes built up in this way will not only be differentiated from each other but will also be retrieved more readily when needed. Attached to a meaning that is seen as part of a whole view of the subject, accessing processes is a matter of sensing the fundamental meaning, then retrieving that particular aspect of mathematical knowledge rather than floundering through a vast array of individual facts and techniques.

Thus, understanding is central to mathematical learning:

- It allows knowledge to be transformed to match novel problem situations.
- It provides more efficient and effective learning.
- It assists with retrieval of facts and processes.

Just as mathematical ways of thinking that will be needed to deal with our increasingly complex society will be different, so too will the teaching that will help to prepare for it require changes. An ability to adapt to these new ways requires understanding over skill. Thus, teaching must move from instruction on how to perform and apply techniques, to focus on building the meaning and understanding that will enable students to examine new situations, and to develop and apply new processes. A teacher's role will be to assist learning, recognising that students construct their own ways of knowing, rather than telling students what to learn. Classrooms will focus on the types of cooperative learning and doing that match the world of business and science, rather than encouraging solely individual learning and assessment.

Problem solving will be central to learning at all stages, from the beginning concepts through the development of sophisticated processes, instead of being seen as the end point of instruction, to be applied to problems resembling the way the content has been taught. It is problem solving that allows concepts, processes and their uses to be built up, problems to be explored and solved, conjectures to be made and examined, and complex ideas about the world to be communicated in precise and concise ways. A problem solving context can mimic real life situations and enable students to see the nature and purpose of the mathematics rather than treating it as an end in itself.

Problem solving is, however, more than a means to teach and develop mathematical knowledge. Being a good problem solver is essential in today's society, where knowing the rules to follow to obtain a correct answer is no longer enough. Indeed, being a good problem solver can be a great advantage in the workplace and in certain circumstances may be more important than having skills, knowledge and experience. Students of today will in all likelihood have several jobs, and even careers, within their lifetime. Being able to construct, interpret, formulate and investigate situations, as well as communicate solutions, will be a distinct advantage when moving into new fields of work.

Students who can think mathematically, analyse problem situations, explore various means to achieve a solution, as well as carry out a plan to solve mathematical problems, acquire deeper and more useful knowledge than simply being able to complete calculations, name shapes, use formulas to make measurements, or determine measures of statistics and probability.

Chapter 1

Approaches to mathematics teaching and learning

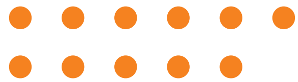
People are more likely to continue learning and using mathematics if they learn it with understanding and see its beauty and the possibility of applying it to matters that interest them, including games as well as more practical matters. Skill with the traditional basics is important to facilitate creative thinking about complex questions. However, skill alone is unlikely to prepare students for their future.

Those we have learned with understanding are more likely to remember the skills, to apply them efficiently and to be able to rediscover skills they may forget. They will be able to transfer their knowledge to new problems in the future and figure out mathematics for new situations. **(Willoughby, 2010, p. 83)**

CONCEPTIONS OF HOW mathematics is learned have changed. Once, the way in which the content was organised was considered paramount and good teaching focused on ways of transmitting this preformed knowledge from teacher to learner. Material was categorised into a detailed syllabus, suggesting that an existing mathematics simply needed to be conveyed to children. Examples of each new idea were produced by the teacher, explained to the class, and followed up with practice in the form of worksheets or textbook pages. The role of the learner was to practise what was provided until it could be readily reproduced. Only then would the (successful) learner be shown and given practice in ways of applying this knowledge to different situations. In turn, the degree to which these procedures could be acquired and used determined the mathematical status of the individual learner and revealed the mathematical aptitudes with which he or she had been endowed.

While an appropriately organised curriculum is always important, teaching is now more learner centred than content driven. In the first instance, this means that the organisation of materials to be taught needs to derive as much from an understanding of how children learn as from the structure of the knowledge to be gained. New ideas and ways of thinking need to be linked to well understood existing knowledge, and their development needs to proceed in ways that reflect the manner in which the learner sees and makes sense of them rather than in the order seen by someone who already has this understanding. It also reflects a realisation that children do not simply take in mathematical knowledge that is merely transmitted to them, no matter how well organised and justified it is. Children are frequently observed to build their own ways of doing mathematics despite material or procedures introduced by a teacher. Sometimes this has led to alternative ways of coming to terms with mathematics and of using it to solve problems:

Asked how many counters there are,



a young child might answer 'Ten and 1 more is 11', or '4 and 4 is eight, 9, 10, 11', '2, 4, 6, 8, 10, 11', or other ways of grouping the counters rather than simply count '1, 2, 3, ... 11'.

At other times, the way that children build their own ways of doing mathematics has produced consistent patterns of errors or misconceptions when place value is overlooked, renaming is neglected, and the significance of zero is not appreciated:

$$\begin{array}{r} 4\ 7 \\ +\ 3\ 9 \\ \hline 7\,16 \end{array} \qquad \begin{array}{r} 6\ 4 \\ -\ 3\ 9 \\ \hline 3\ 5 \end{array} \qquad \begin{array}{r} 4\ \ 9 \\ \times\ 4\ \ 7 \\ \hline 16\ 63 \end{array} \qquad \begin{array}{r} 6\ \ \ 3 \\ 9\overline{)5\ 4\ 2\ 7} \end{array}$$

Learning that builds on the needs and knowledge of individual students also parallels the way in which mathematics evolved as people tried to come to terms with and make sense of problems in their everyday lives. In the early years of school, concepts in number, measurement, geometry, statistics and probability have always been developed in a similar manner through story situations from the children's own lives. Thus, addition and subtraction concepts and facts have grown from realistic embodiments rather than through exercises in acquiring the addition and subtraction symbols + and – and their use with number symbols. This emphasis on problem situations out of which mathematics can grow is essential all the way through a student's schooling. As problems are understood and reconciled, the mathematics that is needed and that can develop from making sense of the solution process is personally developed and owned.

Learning mathematics, then, is necessarily an active process; the concepts and processes are too complex and the ideas often too abstract to allow them to be simply accepted through reading or telling. Children need to be involved in the formation of these new ways of thinking if they are to find them personally meaningful and be able to use them in different settings and formats. Experiences with problematic situations are fundamental to the way in which concepts and processes are built up or acquired, and resources for assisting learning need to incorporate play, games, everyday situations and objects from the child's world as well as specialised materials that might be seen to embody mathematical ideas.

Learning is also a social activity, and both the mathematics and the manner in which it is learned are influenced by the way children interact with each other and with their teachers. Children construct meaning not only through the experiences they have with materials and problems, but also through examining and reflecting on their own reasoning and the reasoning of others. Talking about ideas that are being generated, sharing ways of tackling tasks and resolving difficulties, and describing outcomes that arise are integral to learning mathematics.

Rather than working in isolation from other learners, it is often better for children to work cooperatively so as to encourage mathematical discussion and resolution of the various interpretations that emerge. Group activities and projects need to be organised to allow children to work on shared tasks, rather than have them perform individually on problems, worksheets, even pages from a text. In this way, they can work together in pairs or groups of 3 or 4, taking turns to record any working or observations in order to discuss them with the larger group later. Of course, at first one child may do more of the thinking and activity than the others, but whole class discussion about the mathematics of the situations on which they are working can then focus the need of all children to be able to talk about the activity. Indeed, judicious questioning of the one who watched more than participated can draw that child's attention to the need to attend to all aspects of the task at hand and to voice uncertainties as they occur, rather than leave it to the more capable or dominant child. It will also allow a variety of ways of thinking about a particular situation to arise and add to the richness of the learning for all participants.

Cooperative learning can then go beyond merely working together on set tasks, and an atmosphere can be created in which children construct their own mathematical conceptions. The goals of learning, the discussions about the means of achieving those goals and the individual paths taken need to be at the centre of classroom learning. An attitude that each individual will reach his or her learning goal only if the others in the group also reach their goals is as important as the goals themselves. This is in distinct contrast to a classroom where learning is individualistic or competitive, with children working by themselves at their own pace to achieve goals unrelated to those of their classmates.

Cooperative learning is also crucial in promoting children's ability to communicate and reason mathematically. The interactions between teachers and children, and especially among children, influence what is learned and how it is learned. In particular, attempts to communicate their thinking help to develop children's understanding that mathematics is conjectural in nature—that mathematical activity is concerned with reasoning about possibilities, rather than learning the results presented by others. Indeed, trying to make sense of methods and explanations they see or hear from others is fundamental to constructing mathematical meanings (Yackel et al., 1990). Within a framework of learning cooperatively, each child can be viewed both as an active reorganiser of his or her personal mathematical experiences and as a member of a community or group in which he or she actively participates in the continual regeneration of 'taken-as-shared' ways of doing mathematics (Cobb & Bauersfeld, 1995). Institutionalised practices such as using tens and ones in a place value sense, or following a particular method for measuring the area of a circle, can then emerge anew for each child, yet conform to accepted norms of mathematical behaviour.

This form of learning can lead children to value persistence in working at a challenging task, in contrast to the mere repetition of similar exercises; to engage in meaningful activity in preference to procedures acquired by rote; and to see that cooperation and negotiation are productive at both a personal and social level. Consequently, the learning of mathematics can be viewed as an active, problem solving process in which social interactions help to promote understanding and reconcile the various interpretations and ways of thinking and acting that can arise.

Instructional games and mathematics learning

Outside the classroom, the effort applied in mastery of play activities focused attention on the potential of games for involving children in mathematics. First, because children 'construct much of their reality through playing' (Steffe & Wiegel, 1994, p. 117), and their games almost always involve sustained attention, high level thinking, and collective as well as individual effort. Second, the rule governed behaviour in these games is suggestive of the actions envisaged in the teaching and learning of mathematics. The match between the expectations for involvement in mathematical learning and the behaviours freely committed to game playing led to the development of instructional games which could be included in mathematics programs. Initially this concentrated on practice aspects such as that required for basic fact learning, to provide motivation or to reward children for progress they have made. However, observations of the games in use led to their extension to a wider range of concepts and processes (Larouche, Bergeron & Herscovics, 1984), and use can also be made of the interest generated by games to assist children to generalise to the more abstract recorded forms and higher level mathematical ideas. Vygotsky (1978) has argued that 'the influence of play on a child's development is enormous', in that action and meaning can become separated and abstract thinking can thereby begin.

As learners participate in the playing of instructional games, the manipulation of materials and the verbalisation of actions, thoughts and interpretations assist in the construction of mathematical concepts. An element of chance ensures that each player has an opportunity to win and build self esteem. Games themselves are seen as fun, not only providing motivation but also ensuring the full engagement on which constructive learning depends (Blum & Yocom, 1996). Often this means that while children may not engage in learning to please a parent or teacher, and can rarely accept that mathematics will be useful in later life, they will willingly learn in order to participate with their peers in socially rewarding activities. There are many computer games that provide this type of learning situation and motivation (Booker, 2002; 2004).

Games should not be regarded merely as a useful activity when teaching and learning have been accomplished, but should be seen as an integral part of a balanced program. They are most effective when structured around mathematical ideas, and when playing is dependent on mathematical understanding. In this way, they provide a context that is real to children as they become fully engaged in something in which the outcome matters, leading to a realisation of the value of the underlying mathematical processes. Indeed, more general notions such as predicting, testing, conjecturing, generalising, justifying and checking are often integral to the purpose of winning a game (Ainley, 1990). This same purpose has children willingly trying out new ideas which they must justify in a meaningful way to those against whom they are playing (Ernest, 1986). There is also a strong incentive to check or challenge the mathematics of other players during games when there is a chance of winning or being defeated, providing a meaningful context for discussion and a need for clear communication.

When they are integral to teaching, games also more readily allow a teacher to make judgements about student understanding. Ways of thinking become much more apparent during playing, not only as ideas are made public, but also through the actions which reveal underlying thinking. Within a game, children are not endeavouring to provide an answer or reason which they think will match a teacher's expectations, but focus on those that make sense to them. Close observation of small groups provides much greater insight into students' ways of knowing and allows the development of more appropriate and effective mathematics programs. Feedback from peers is provided immediately, compared to that with teacher directed instruction and worksheets where the responses to a misconception are often so delayed that the thinking that resulted in a particular answer can no longer be remembered. In these ways, games also have an important contribution to make to the shift in assessment from judgement of student output to portrayal of capabilities.

Because of these many contributions to the development of knowledge and positive influences on the effective components of learning situations, instructional games have always featured in mathematics teaching and learning in the early years. In particular, they have been used to develop and construct concepts, to build, maintain and consolidate skills such as basic fact mastery, and to improve problem solving abilities and strategic thinking. However, much of this use has been to provide additional practice on mathematical ideas or processes introduced in prior teaching. Games should also be used to put forward new notions, to lay the foundations for concepts and processes that would be formalised later, and to allow discussion on possible generalisations from earlier knowledge prior to any formal introduction of new concepts or processes.

For these social and cognitive reasons, games have an important place in mathematics education (Booker, 2000). They contribute to teaching and learning by providing a background in which mathematical concepts can be developed and constructed. Problem solving ability is improved when the discovery and use of strategies is required and previously acquired skills are maintained through motivating practice. Reference will be made to each of these aspects of game use in Section 2 of this book, where a variety of instructional games will be introduced. At the same time, the social interactions conducive to learning have also been borne in mind so that methods of play requiring cooperation have been deliberately invoked. In this way, children learn that without cooperation a game may not proceed and there will certainly not be any chance of winning. Listening to other players, talking about what is happening, and even assisting others to understand and complete the tasks involved in the game need to be seen to be critical playing behaviours. Children then learn from one another, as much as from the structured activities, through sharing of the method of play, consequences and needs of the game. In turn, the act of winning, or even almost winning, produces a desire for further play and thus further experience with the underlying concept or skill.

Above all, involvement in instructional games induces children to make sense of their ideas and the interpretations of others. The dialogue engaged in while playing facilitates the construction of mathematical knowledge, allowing the articulation and manipulation of each player's thinking.

Such communication helps to extend a conceptual framework through a process of reflection and points to the central role of language, as the social interaction gives rise to genuine mathematical issues. In turn, these problems engender an exchange of ideas, with children striving to make sense of their mathematical activity, and lead them to see mathematics as a social process of sense making requiring the construction of consensual mathematical understandings.

NUMERACY

> *To be truly numerate … students need to learn mathematics in ways that enable them to recognize when mathematics might help them to interpret information or solve practical problems, apply their knowledge appropriately in contexts where they will have to use their mathematical reasoning processes, choose mathematics that makes sense in the circumstances, make assumptions, resolve ambiguity and judge what is reasonable.* (National Numeracy Review Report, DEEWR, 2007, p. xi)

The nature of what is to be studied has also changed. As technology becomes ever more central to all aspects of life, there are increasing calls for a more numerate population. It is no longer considered sufficient for children simply to study mathematics; they need to be able to *use* their mathematical knowledge in an ever broadening range of activities. As Orrill (2001) remarks, we are living in a society ‘awash in numbers’ and ‘drenched in data’. For those comfortable with and competent in thinking about numbers, this provides a basis to evaluate such issues as the benefits and risks of medical treatments, estimates for budgets that will allow or disallow access to education and transport, and many other concerns that were once only available to specialists and those in the know. Conversely, individuals who lack an ability to think numerically will be disadvantaged and at the mercy of other people’s interpretation and manipulation of numbers. Indeed, ‘an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg’s time’ (Steen, 1997). In contrast to the study of further mathematics, numeracy is concerned with applying elementary ideas in sophisticated settings, rather than generalising these ideas to more abstract concepts and more complex processes.

> *Mathematics thrives as a discipline and as a school subject because it was (and still is) the tool par excellence for comprehending ideas of the scientific age. Numeracy will thrive similarly because it is the natural tool for comprehending information in the computer age.* (Steen, 2001, p. 111)

The term ‘numeracy’, concerned with using, communicating and making sense of mathematics in a range of everyday applications, emerged to provide a more satisfactory description of these extended mathematical processes and ways of thinking. Initially proposed as a ‘mirror image’ to literacy (Crowther, 1959) related to relatively advanced mathematics, comparison with ‘reading and writing’ has frequently led to its interpretation as the skills associated with simple number computation. However, the mathematics used in everyday life has always meant more than the manipulation of numbers. Just as the meaning of literacy has broadened to incorporate the integration of reading, writing, listening, speaking and critical thinking, and to recognise the importance of context in the making of meaning, its correlate numeracy (sometimes referred to as mathematical or quantitative literacy) now extends to an ability to explore, conjecture and reason logically, and to use a variety of mathematical methods to solve problems (National Council of Teachers of Mathematics, 1989; 2000).

Sense making, such as number sense, which was first referred to in the context of numeracy as ‘at homeness with numbers’, is also fundamental, along with the need for ‘an appreciation and understanding of information presented in mathematical terms in order to use and apply mathematical skills and to communicate mathematically’ (Cockcroft, 1982). Positive attitudes towards *involvement* in mathematics, problem solving and applications, as well as a capacity to work systematically and logically, and to communicate with and about mathematics, are also central to being numerate. Mathematical communication abilities are essential in understanding and assessing the proposals of others, to convey arguments and justifications to a broader audience, and to analyse and interpret information. Discussing and writing about the mathematics that has been completed, and focusing on the thinking processes that were followed, the attempts that were made along the way, and the justification that allows particular approaches and solutions, can be used to build up an ability to read, write and speak with and about mathematics.

As the certainties of the past have given way to the uncertainties of the present and future, this also means that the formal techniques of number and geometry that gave exact and unalterable results must make room for ways to examine and explore less certain situations using statistics and probability, and include a range of estimation and approximation processes. Thus, numeracy should be seen to include the content of mathematics, particularly number sense, spatial sense, measurement sense, data sense and a feeling for chance, together with a focus on problem solving and the uses of mathematics in communication, as shown in the following table.

NUMERACY

CONTENT	+ PROBLEM SOLVING	+ SENSE MAKING	+ COMMUNICATION
• number • measurement • geometry • statistics and probability • algebra *Focus on* • technology—calculators, computers as tools to aid thinking • estimation vs exact • mental processes as well as recorded	• analyse the problem • explore possible means to a solution • select and try a solution process • analyse solution and possible answer’s sense in problem context	*number sense* • understanding of numeration and computation *spatial sense* • visualisation of properties and relationships *measurement sense* • application of number and spatial sense *sense of statistics and probability* • inclination to use flexibility • personal strategies • ability to interpret	• discuss and write about mathematics • reflection • present arguments in mathematical form • interpret data presented graphically and statistically

The Australian Association of Mathematics Teachers was among the first to attempt to provide a working definition of numeracy to assist teachers in their planning and teaching:

> *To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid* [sic] *work, and for participation in community and civic life.*
>
> *In school education, numeracy is a fundamental component of learning, performance, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:*
>
> - *underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic)*
> - *mathematical thinking and strategies*
> - *general thinking skills*
> - *grounded appreciation of context.* (Australian Association of Mathematics Teachers, 1997, p. 15)

Building on the initial notions of numeracy that first evolved in the United Kingdom (Crowther, 1959; Cockcroft, 1982), the Department for Education and Employment formulated a concise statement of what was required as part of the new national curriculum:

> *An understanding of the number system, a repertoire of computational skills and an inclination and ability to solve number problems in a variety of contexts. Numeracy also demands practical understanding of the ways in which information is gathered by counting, measuring, and is presented in graphs, diagrams, charts and tables.* (DfEE, 1998, p. 11)

Further general statements were proposed for Europe (OECD, 2004) and Australia (DETYA, 2000):

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to use and engage with mathematics in ways that meet the needs of the individual's life as a constructive, concerned and reflective citizen.* (OECD, 2004, p. 15)
>
> *Numeracy provides key enabling skills essential for achieving success in schooling. Sound numeracy skills acquired in schooling support effective participation in personal, economic and civic contexts. The increase of globalisation and the use of technology have generated increased demands for a more numerate Australia.* (DETYA, 2000, p. 1)

The national *Numeracy Benchmarks* (Curriculum Corporation, 2000) articulated important and essential elements of numeracy at nationally agreed minimum acceptable levels, although they did not attempt to describe the whole of numeracy learning, nor the full range of what students needed to be taught. They noted that numeracy must be seen as only a part of mathematics, and that students should attain high standards of knowledge, skills and understanding through a comprehensive and balanced curriculum. Consequently, these benchmarks have been defined as minimum standards without which a student would have difficulty progressing at school and do not describe proficiency in numeracy or even the minimum standards that the community expects (National Numeracy Review Report, DEEWR, 2007, p. 14). Beginning in 2008, agreed national testing replaced the various state and territory assessment programs (NAP, 2012) to determine the proportion of students who have achieved nationally agreed standards. Eventually, this testing will reflect the aims of a national curriculum applying to all school systems. Indeed, the *Australian Curriculum: Mathematics* incorporates conceptual understanding, fluency, problem solving and reasoning as its core standards and expectations (Proficiency strands, *Australian Curriculum: Mathematics*, Australian Curriculum and Assessment Reporting Authority (ACARA), 2012):

- *Understanding*—comprehension of mathematical concepts, operations and relations, and the connections among them—the 'why' as well as the 'how' of mathematics.
- *Fluency*—facility in carrying out processes flexibly, accurately, efficiently and appropriately; ready recall of factual knowledge.
- *Problem solving*—ability to formulate, represent and solve mathematical problems, and to communicate solutions effectively.
- *Reasoning*—capacity for logical thought, reflection, explanation, justification and proof—exploration, generalisation, description.

These four strands are then being used to underpin the three content strands—*Number and Algebra*, *Measurement and Geometry*, *Statistics and Probability*—that will make up the curriculum content, with *Working Mathematically* as an overarching approach:

MATHEMATICS—WORKING MATHEMATICALLY			
	NUMBER AND ALGEBRA	**MEASUREMENT AND GEOMETRY**	**STATISTICS AND PROBABILITY**
Understanding	numeration additive multiplicative algorithmic algebraic	units of measurement shape geometric reasoning location and transformation	chance data reprersentation and interpretation
Fluency			
Problem solving			
Reasoning			

This summary of the components of the *Australian Curriculum: Mathematics* framework has been designed to highlight several important aspects that need to be taken into account. The relevant space in the diagram given to each proficiency strand is designed to show the following:

- The development of conceptual understanding needs to be established prior to further development of a topic or content area, as it is critical to building up fluency and problem solving.

- Problem solving does not arise simply as a consequence of understanding and fluency. As much time and resources need to be allocated to establishing problem solving as to building competence with computational and other processes.

- Reasoning will be developed along with understanding, fluency and problem solving, but there also needs to be time devoted to ensuring that students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising.

Within Number and Algebra, *Numeration* (understanding number) is critical; addition and subtraction are grouped as *Additive thinking*; multiplication and division are brought together as *Multiplicative thinking*; and emphasis on the thinking underpinning numeration and computational processes is viewed as *Algorithmic thinking*, a way of thinking that underpins all higher mathematics, rather than simply as procedures for obtaining answers that can be learned in isolation. Similarly, the components of Measurement, Geometry, Statistics and Probability, drawn from the content descriptions, need to be seen as linked together with Number and Algebra to form a connected and coherent view of the mathematics to be learned, rather than as distinct topics each with different procedures and ways of operating.

As Hiebert and Grouws (2007) noted in their synthesis of international research calling for a more detailed, richer and coherent knowledge base to inform practice:

> *Two features of classroom teaching facilitate students' conceptual development (and mathematical proficiency), explicit attention to connections among ideas, facts and processes, and engagement of students struggling with mathematics.* (Hiebert & Grouws, 2007, p. 391)

The *Australian Curriculum: Mathematics* has been designed to build students' conceptual understanding, and thus fluency, by attending explicitly to the connections among all aspects of the mathematical content. As teachers, we need to ensure that students engage with the mathematics that they are learning and that this mathematics is not simply focused on routines that are readily known, but provides a move towards deeper and more powerful knowledge that provides a challenge to all students at all levels. Struggling to come to terms with this mathematics will then become a natural and enjoyable part of mathematics learning.

CONSTRUCTING MATHEMATICAL CONCEPTS AND PROCESSES

While the new emphasis on teaching the mathematics that has traditionally been the mainstay of the primary school has been on the development of numeracy, it has also been argued by Ma (1999) that an understanding of elementary mathematical ideas essentially underpins the development of all mathematics. She argues that the early mathematics of number and space is *fundamental* in that all of the new branches, whether pure or applied, from measurement, through trigonometry to calculus and beyond, have developed from basic concepts and processes established with numerical and spatial reasoning. Second, it is *primary*, containing the rudiments of many important concepts needed in more advanced topics. This has profound implications for teaching, as Ma notes in quoting Chinese teachers' wisdom on the development of early mathematical ideas:

> *If students learn a concept thoroughly the first time it is introduced, one will get twice the result from half the effort. Otherwise, one will get half the result with twice the effort.* (Ma, 1999, p. 115)

Third, it is *elementary* because it is the beginning of students' learning and appears straightforward and easy, yet provides a basis for later generalisations. Rather than merely being a simple collection of disconnected number facts and computational algorithms, numeracy establishes the basis on which future mathematical thinking is constructed.

This perspective, that mathematics is learned by individuals constructing ideas, processes and understanding for themselves rather than through the transmission of preformed knowledge from teacher to learner, now dominates conceptions of mathematics learning (Hennessey, Higley & Chesnut, 2012; Lambdin & Walcott, 2007, p. 15; Goldin, 2002, p. 204; Simon, 1995; Malone & Taylor, 1993). The view of learning known as *constructivism* considers that:

- knowledge is actively created or invented, not passively received
- new ways of knowing are built through reflection on physical and mental actions
- learning is a social process requiring engagement in dialogue, discussion, argumentation and negotiation of meanings.

Evidence for a constructivist epistemology began to surface as errors that children made were analysed and the misconceptions that they had formed were seen to fall into common patterns rather than to reflect individual inconsistencies. Writing teen numbers back to front—for example, '41' for fourteen—occurs because children generalise the pattern for other 2 digit numbers and write the digits in the order in which they hear them. Misinterpreting numbers with zeros (such as seeing 3005 as three hundred and five based on its symbolic writing, rather than as three thousand and five as place value notions suggest), carrying out computational procedures from left to right in the same way that numbers are read, subtracting the smaller number from the larger or dividing the larger number by the smaller (so that $3 \div 6$ is given as 2) all show how children are able to construct their own ways of interpreting and carrying out mathematical processes.

The pervasiveness of these patterns of thinking from child to child and from situation to situation despite well structured and well presented teaching forced an awareness that there was something compelling individual children to build their own view of mathematics. It was not so much that the children were mistaken, but that their alternative was incomplete—reasonable from their perspective, but inappropriate or inconsistent from the viewpoint of mathematics. Their errors did not arise carelessly, but deliberately, as they set out to obtain answers using thinking that they felt would get them there. This realisation that children were able to construct their own set of *explanations* for their mathematics led to a reconception of what was happening as mathematical ideas and processes were learned. If children were able to invent their own methods and explanations that showed up in patterns of errors, they would also be building their own understanding of appropriate ways of thinking rather than simply taking in a teacher's explanations. Thus, students came to be viewed as inevitably performing constructions, 'some flimsy and indistinguishable from rote learning, some powerful and highly generative', but leaving teachers unsure of 'the kinds of constructions students made when [they] demonstrate a technique by showing or produce a solution by telling' (Noddings, 1993, p. 38).

From these observations, the approach to teaching and learning mathematics known as constructivism was proposed. In essence, this approach acknowledges that mathematical knowledge is a product of an individual's mental acts. It is a way of seeing, of organising experience; a set of mental constructs that are used to view the world (Confrey, 1990). Such a view suggests that the major role of the teacher is to help children create more powerful constructions, and that developing autonomy and self motivation is vital (Clements, 1997). In this way, the classroom is transformed from a place where teachers attempt to gain the desired performance by children to a place where children are seeking out challenges, attempting to solve them, negotiating paths to those solutions with the teacher and other children, and then reflecting on what has happened.

An essential feature of this view is that existing conceptions, whether gained from everyday experiences or previous learning, guide the understanding and interpretation of any new information or situation that is met. This often results in resistance to adopting new forms of knowledge or an unwillingness to give up or adapt previously successful thinking. Indeed, old ways of thinking are not usually given up without resistance, and their replacement by, or extension to, new conceptions is guided by those that already exist. Consequently, the intuitive beliefs and methods of children may appear very different from accepted mathematical practice and may also be very resistant to change (Brousseau, 1997). Teachers will need to ask questions that challenge ill-formed ideas and inappropriate generalisations, and pose new problems that will require the revision of old constructions or ways of thinking. The role of a teacher in a constructivist classroom is thus more one of a guide or mentor than a director of what needs to be done in order to become proficient, although there will still be a place for showing how things are to be done. The challenge will then be to lead children to come to understand and accept this as a method of their own, rather than simply practising and acquiring by rote another person's way of doing something. Evidence from children who have experienced difficulty in learning mathematics has also shown that those who simply acquired teacher taught techniques by rote were often unable to apply this knowledge and frequently forgot, or at least were unable to recall when needed, knowledge that had been earlier assumed to have been learned (Booker, 2011). In contrast, those who participated actively in their own learning were more able to use this knowledge and tended to maintain it for future use and adaptation. As the novelist Isabel Allende put it in *Portrait in Sepia*:

> *Every time I asked a question, that magnificent teacher, instead of giving the answer, showed me how to find it. She taught me to organise my thoughts, to do research, to read and listen, to seek alternatives, to resolve old problems with new solutions, to argue logically. Above all, she taught me not to believe anything blindly, to doubt, and to question even what seemed irrefutably true.* (2001, p. 151)

A constructivist perspective on the teaching and learning of mathematics, then, focuses on the learner, setting out to guide her or him in the construction of mathematical ways of knowing and operating based on existing knowledge. This requires three phases (Herscovics & Bergeron, 1984):

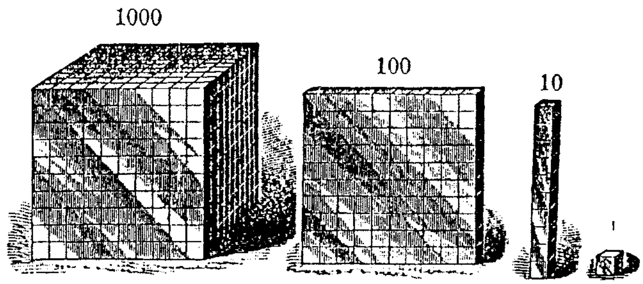
1 Determine the form of knowledge that may be used as a foundation for building the intended concept or process.
2 Ascertain whether such a basis is present for each learner.
3 Ensure that each step in the proposed construction is accessible to each individual.

Under the guidance of a competent teacher, mathematics learning can then be viewed as a process of reconstructing particular ways of thinking, rather than as reinventing pre-existing mathematical thoughts. When children construct their own mathematics, that knowledge is both personal and owned; something over which they have control so that their learning experiences empower them rather than leave them relying on procedures that have been developed by someone unknown, in response to problems that are no longer remembered, from a time and situation that no one can recall.

This contrasts with more traditional approaches to teaching using a 'good definition' to transmit a given concept and a suitable standardised procedure to provide a mathematical process. This form of teaching is still paramount in many classrooms—not necessarily by design or even intent, but through the premature introduction of concise definitions, formal symbolism or technical terminology. Pressures to cover a specific program in a given time may prompt a teacher to choose such an approach, as it seems more efficient in the short term. However, teachers' familiarity with the mathematics and their own formal ways of thinking often cause them to overlook the obstacles that the child must overcome in building this cognition. If the mathematics to be introduced cannot be related to the child's experiences, it simply will not make sense and the child will be reduced to manipulating meaningless symbols using rules that are not understood.

Many difficulties in learning mathematics can thus be ascribed to discrepancies in the type of activities engaged in or the timing of their introduction. Consequently, an essential part of teaching mathematics is to provide meaningful experiences at appropriate points out of which appreciation and understanding of concepts and ways of thinking can be built. Such activities can be meaningful in the teaching of mathematics only when two interrelated aspects are present. The meaning inherent in the situation, materials, patterns, language or symbols through which the new notions are being expressed must match the mathematical meaning that is intended to be built up. These meanings must also match the children's level of development and ability to take in information, to generalise and construct a reasonable view of the underlying mathematics. In other words, experiences need to be meaningful for both the mathematics and the child.

Materials in use today to represent numbers, such as ice cream sticks bundled to show tens and ones, not only provide a structure that the children can see, but also match identically the behaviour of the number system itself. Such materials have a long history, from the use of bead frames to show tens and hundreds popularised by Maria Montessori in the early 1900s and the use of blocks to show the ones, tens, hundreds and thousands promoted in arithmetic books of the 19th century.



Fish's New Arithmetical Series 1883

Nonetheless, as Cobb, Yackel and Wood observed, 'manipulative materials can play a central role if we wish students to learn with understanding, but the way the materials are interpreted and acted upon must necessarily be negotiated by the teacher and students (1992, p. 7). The materials are not 'transparent' representations of a readily apprehensible mathematics, but instead are vehicles for the potential meanings that children might construct.

USING MATERIALS IN MATHEMATICS

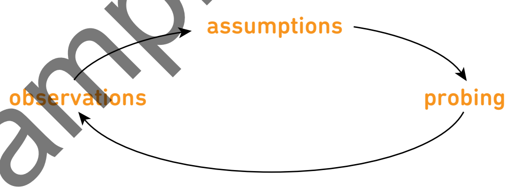
The need for materials is fundamental in teaching mathematics because so many of the ideas that have to be learned are not intrinsically obvious. They were generalised and developed from diverse and obscure situations over a long period of time, usually by mature thinkers who had particular social or intellectual needs. If young children are to be assisted to develop the same forms of thinking, then situations in which these ideas can be discerned and discussed are essential. It is for this reason that the teaching of mathematics has had a long tradition of using structured materials, materials through which the underlying mathematical ideas might come to be perceived and appreciated. For instance, ten frames and bundling sticks are used to establish early numbers, Base 10 materials show the place value patterns for larger numbers, and region models give meaning to fraction ideas. Even comparatively simple notions such as the initial number concepts are abstractions rather than something that can be seen in the immediate environment. The number four does not exist in the real world, but is a representation that the child needs to construct for himself or herself by linking objects with language and symbols within meaningful experiences with materials that sometimes show and sometimes do not show the concept of *fourness*. Similarly, the basic spatial forms can only take on meaning through seeing and making representations of the general notion of what will come to be called a rectangle or a triangle and learning to distinguish one from the other.

At the same time, it must be borne in mind that materials by themselves do not literally carry mathematical meaning. While they might assist in the initial building of understanding, it is reflection on the actions of the materials and the situations that they represent that allows the generalisation to a mathematical way of thinking, rather than a rote learning of the results of these actions. If children focus solely on the outcomes, it is possible that they will simply learn at a surface level how to manipulate the materials rather than the deeper, fundamental mathematical ideas. This risk becomes even larger if the materials come to be seen as ends in themselves, rather than as links to the mathematical concepts and processes they are intended to represent.

where others have gained competence, or to identify broad achievement levels for a new class member on transfer from another school. It is often the case, for example, that children have difficulties with written algorithms because of undetected misconceptions in numeration concepts or because they use inefficient methods for obtaining number facts. Indeed, any assumption about a student's capacity has to be checked to see if it can be confirmed. In the midst of teaching, it is easy to make quick conjectures with only one possibility being considered and then to act by simply telling or restating a strategy that seems appropriate from the teacher's perspective. At best, this might make a child very dependent on a teacher interceding to tell him or her just what to do. Just as likely, it will discourage self confidence and induce the child to do the minimum amount of work possible so as to avoid being seen to be having difficulties with work that other children do not find taxing.

Instead, what is needed is to accept a student's response at face value and then provide a similar problem or exercise so that a further observation can be made, this time equipped with an expectation that something is not quite as it ought to be. An observation is now more focused on the particular misconception or faulty process and is likely to provide further information on the source of the difficulty. Armed with one or more possibilities as to the cause, a teacher can then systematically account for each one with well chosen questions to pinpoint the underlying misconceptions that need attention. Without this indepth probing, only a surface level response can be made and neither the teacher nor the student will have any confidence that the same difficulty will not occur again soon after.

Diagnostic assessment can thus be summarised as a 3 phase cycle (Booker, 2011), where a series of observations, assumptions and probing are followed until the underlying causes for any underlying difficulties are revealed:



This cycle of assessment can underpin a teacher's ability to plan for individual students, review a proposed sequence of work or activities, and determine whether to move on to new ideas or applications or to step back to where difficulties are arising for a student or group of students.

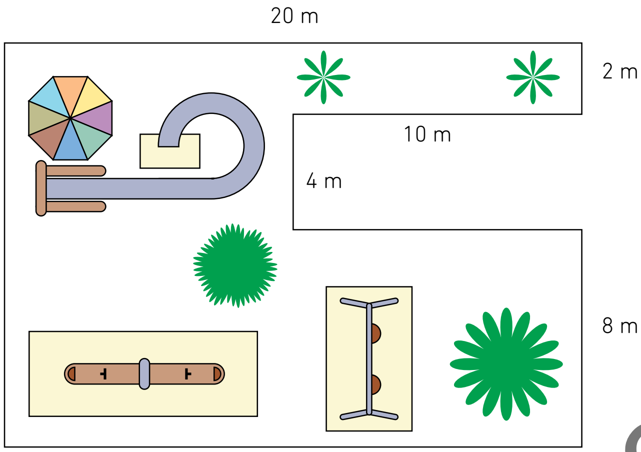
National Assessment Program in Numeracy (NAPLAN)

In addition to the assessment practices that teachers use to plan their teaching, there are now programs of assessment that are overseen by ACARA. The National Assessment Program of numeracy tests was part of the first set of national testing *National Assessment Program in Literacy and Numeracy*, commonly known as NAPLAN, which began in 2008. The *National Assessment in Numeracy* tests are given in May for students in Years 3, 5, 7 and 9 in all Australian states and territories in a similar manner to state and national testing in countries such as the United States, the United Kingdom and New Zealand.

NAPLAN tests consist of many multiple choice items, with some short answer questions, and in the later years, take two forms—one where a calculator can be used and one where it cannot.

This is the plan of a playground.

What is the perimeter of the playground?



44 m	78 m	80 m	88 m
◯	◯	◯	◯

Shade one bubble.

Adapted from a Year 5 numeracy question that requires the bubble below the correct answer to be shaded.

Rachel has a moneybox with some $1 and $2 coins.

The total mass of the coins is 106.8 grams.

Rachel knows that:

- the mass of a $1 coin is 9 grams; and
- the mass of a $2 coin is 6.6 grams.

What is the smallest mass of exactly $5 worth of coins?

Write your answer in the box.

What is the total value of the coins in Rachel's moneybox?

Write your answer in the box.

Adapted from a Year 7 numeracy question that allows the use of a calculator with the answer written in the boxes.

While it is invaluable to obtain data in this way for individual schools, regions and states, since they measure all achievement against a common framework of content and questions, by their nature these tests are unable to probe understanding deeply nor allow for the original thinking involved with problem solving, generalisations of ideas, and alternative ways to achieve likely answers.

If such a test assesses important mathematics in ways that require students to demonstrate mathematical thinking and proficiency, the test might effectively support a comprehensive mathematics program. Students in a well balanced mathematics program anchored in understanding, fluency, problem solving and reasoning (as emphasised in the framework for the *Australian Curriculum: Mathematics*) are likely to do well on these tests with or without special preparation strategies. However, a test based largely on content that can be tested economically in a multiple choice format often encourages students to try out all possible answers to a problem rather than actually solving it (Sealy, 2006).

Nonetheless, the way in which they are reported, using a form of box and whisker plot similar to that used for the international Program for International Student Assessment (PISA, <www.pisa.oecd.org>, De Lange, 2007) tests, allows 'both the status of, and gain in, individual student achievement to be monitored and reported throughout each student's years of schooling'. In addition, the use of a common scale, divided into ten bands of increasing complexity that spans Year 3, 5, 7 and 9, 'allows both the status of, and gain in, individual student achievement to be monitored and reported throughout each student's years of schooling' (*Assessment Scales*, NAP, 2012). Further, the tests reflect National Minimum Standards that represent 'increasingly challenging skills and understandings as students move through the years of schooling from Year 3 to Year 9' (*Results and Reports*, NAP, 2012). Students with results in the minimum standard band demonstrate the basic elements of numeracy for a year level but 'require additional assistance to enable them to achieve their potential', while those with results in the lowest band have not achieved the national minimum standard for that year and 'are at risk of being unable to progress satisfactorily at school without targeted intervention' (*National Minimum Standards*, NAP, 2012). Over time, it is hoped that most children will move to the upper three bands within the assessment and thus be more prepared to participate in higher level mathematics.

There are high stakes associated with these tests as funding for education is tied to them to some degree, so both teachers and students are being encouraged to take them very seriously and prepare diligently for them (Wilson, 2007). When this provides a deeper understanding of the concepts, fluent process and problem solving that lies at the heart of mathematics and numeracy, this can be very enriching for children's learning. However, there is always the danger that teachers might 'teach to the test', focusing at a surface level on how to indicate an answer for each multiple choice question and practising for long periods of school time the tests given in earlier years. Not that some form of preparation for test taking should be overlooked—learning how to review significant ideas met over previous years of schooling, reading questions carefully to see what is actually being asked (similar to the problem solving process introduced in Chapter 2), and developing an understanding of time management per question are all essential to performing on many tasks in real life as well as in standardised testing situations. Nonetheless, an excessive focus on the tests *per se* is likely to be unhelpful to student learning, as noted by ACARA chair Barry McGraw in an interview with the Brisbane *Courier Mail*, published on 1 May 2012. McGraw explained that practising NAPLAN exams over and over is not the right way to prepare children for the tests. Instead, the way to prepare for NAPLAN is to give children rich experiences with mathematics. ‘All the students need is to be familiar with the form of the assessment,’ he said.

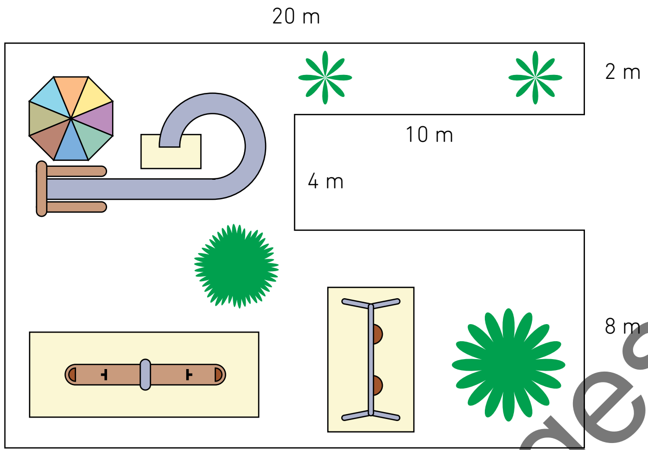
This point was made forcefully by the then President of the National Council of Teachers of Mathematics, Cathy Sealy:

> *Teachers may be expected to ‘set aside’ their mathematics program and instead prepare students for the test. This may mean weeks or even months of missed instructional time. If preparing for the test means practicing a few items to get used to the format, it might serve students well. Too often, however, test preparation also includes learning tricks and tips that may or may not prove helpful on the test. For example, some schools use materials built on ‘clue words’ for solving story problems or teach other tricks about what to do if presented with particular types of problems. Students memorise such phrases and words as all together, more than, and total, associating each with a particular operation. Students are better served by learning the concepts behind the numbers and operations so well that they carry mental pictures of what addition, subtraction, multiplication, or division mean. Recognising a mathematical operation in the context of a problem and knowing how to perform the operation are far better preparation strategies than memorizing tricks or a list of words.*
> (Sealy, 2006)

In other words, excessive time spent practising the test would be better spent ensuring that concepts behind the numbers and operations are so well known that mental representations can be applied to problems where recognising what is needed, and being able to perform processes fluently, are the key to answering NAPLAN questions. Nonetheless, practice on the form of question asked to develop familiarity with the multiple choice format so that a correct answer can be obtained and matched to one of those offered is invaluable. For instance, with the question at the top of page 34, to determine the perimeter of the playground, it would be best to provide the question first with no options to choose from so that students learn to come to terms with what the problem is asking, then formulate a solution and arrive at an answer. The problem can then be presented to the class with 6 possibilities, rather than the usual 4 possibilities, for them to choose among.

This is the plan of a playground.

What is the perimeter of the playground?



I don't know	44 m	78 m	80 m	88 m	Another answer
⬭	⬭	⬭	⬭	⬭	⬭

Shade one bubble.

Each student could indicate their choice so that a range of answers would be given and then each answer discussed in turn, paying as much attention to the incorrect answers as the correct one (80 m). In this way, students would be led to analyse the question to see what was needed and being asked, then determine an answer, rather than simply checking to see which of the given answers is most likely. This approach is of course essential for the questions where there is no indication of the required answer and a student must determine the process and likely answer by applying their own knowledge. For example:

Rachel has a moneybox with some $1 and $2 coins.

The total mass of the coins is 106.8 grams.

Rachel knows that:

- the mass of a $1 coin is 9 grams; and
- the mass of a $2 coin is 6.6 grams.

What is the total value of the coins in Rachel's moneybox?

Write your answer in the box.

In order to answer this question, it must first be noted that a multiple of \$2 coins is needed that will give product ending in '.8'. This can only occur when 6.6 is multiplied by 3 or 8. When 6.6 is multiplied by 3, 19.8 g is obtained but this leaves 87, which is not divisible by 9, the mass of the \$1 coin. When 6.6 is multiplied by 8, 52.8 g is obtained; this leaves 54, which, divided by 9, gives 6. Thus, there would be 8 \$2 coins and 6 \$1 coins, so the value of the coins in her moneybox would be \$22.

Assessment is important in understanding the knowledge that students are constructing, the meanings they give to mathematical ideas, and their developing mathematical ability. Above all, assessment needs to be aligned with teaching so that it supports the overall goals for the learner, the teacher and the mathematics. If the focus and form of assessment are different from those of teaching and learning, then assessment can subvert students' learning by sending conflicting messages about what is valued. On the other hand, assessment that enhances mathematics learning can become an integral part of ongoing classroom activity, encouraging and supporting further learning. Indeed, continuous assessment of students' work and working not only facilitates their learning of mathematics but also enhances their confidence and understanding. As they learn to monitor their own development, they can reflect on their progress, be confident in their understanding and proficiency, and ascertain what they have yet to learn.

Calculators are, however, no more a simple replacement for other ways of doing things than they are obvious to use. Most adults who use calculators at work, home or for their leisure activities were self-taught and consequently feel less than adequately competent with certain functions or able to do certain computations efficiently. Effective ways of using this technology need to occur from the beginning years of school so that it is seen as simply one way among many to determine answers, another way of expressing results, and, above all, as a tool to which understanding and reasoning can be applied. In order to use a calculator in these ways a number of mathematical understandings and skills are required. Recognition of the various symbols and the actions they represent (+, –, ×, ÷, =, √, %) is essential, as is knowing the concepts for these operations and the variety of meanings and situations in which they might be used. Familiarity with the operations in other contexts is necessary in order to know when to use them, to provide some awareness of likely outcomes, and to establish a measure of the reasonableness of results. A plan to 'see' what is happening when the calculator does not show it and for recording intermediate results is also beneficial. It is helpful to learn to use the nonwriting hand to operate the calculator, to the point of using several fingers akin to touch typing, so that the writing hand is available to write any outcome or important points along the way. However, it should be noted that today's children and young adults are much more adept at holding a calculator in one hand, using only a thumb to key in numbers and operations in the same way that they have used electronic games and mobile phones! An awareness of how the various memory functions operate, including the constant keys, allows more efficient computation.

The activities developed throughout this book refer to suitable calculator activities, but the difficulty that always arises relates to the different key formats, operating procedures and capabilities. Familiarity with the calculators used in the classroom needs to be built up by the teacher and then explored jointly with the children.