

**SAMPLE**

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STANDARD LEVEL



# Mathematics

## Applications and Interpretation

for the IB Diploma



Pearson

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### Learning objectives

By the end of this chapter, you should be familiar with...

- modelling linear, linear piecewise, quadratic, cubic, exponential, direct/inverse variation, and trigonometric phenomena
- developing and fitting models (recognising the context, choosing an appropriate model, determining a reasonable domain and range, using technology to find parameters)
- testing and reflecting upon models (commenting on appropriateness and reasonableness of a model, justifying the choice of a model)
- using models (reading, interpreting, and making predictions, avoiding extrapolation).

Mathematical models help us to describe the world around us. In this chapter, we will look at several different kinds of mathematical models. We will examine how to choose, develop, test, apply, and extend a model. Here are some examples of the different kinds of models we see in the world around us:

- On a long flight, the airspeed of a plane is constant, so the distance remaining to the destination can be described by a **linear** model.
- In a situation where the price to manufacture  $x$  units of some product decreases **linearly**, the revenue from selling  $x$  units can be described by a **quadratic** model.
- The volume of a balloon relative to its diameter can be described by a **cubic** model.
- The spread of algae in a polluted lake can be described by an **exponential** model.
- A DJ charges a fixed amount to provide music for a party. The cost is spread equally among everyone who attends the party. The cost per person can be described by an **inverse variation** model.
- The price of electricity is often billed per kilowatt-hour (kWh), so the cost of powering the lights of a stadium relative to the time the lights are on can be described by a **direct variation** model.
- The height of a person above the ground on a Ferris wheel can be described by a **trigonometric** model.

The process of mathematical modelling is a design process and is illustrated in Figure 6.1.

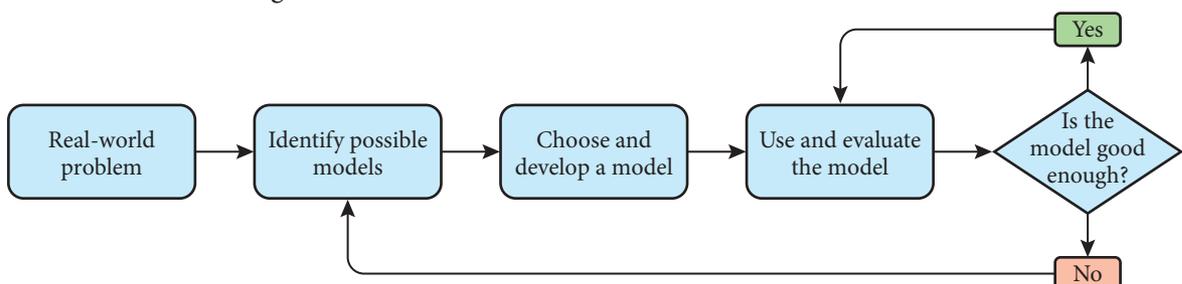


Figure 6.1 Mathematical modelling process

In this chapter we will learn how to develop models both by hand and by using technology. We will discuss how to test a model and then how to reflect upon or analyse the validity of the model. Finally, we will talk about how to use and, if needed, extend or revise a model.

## 6.1 Linear models

**Linear models** are used to describe situations where one quantity (the **dependent variable**) increases at a fixed rate relative to another quantity (the **independent variable**).

### Developing and testing a linear model

Suppose that you need a plumbing repair in your home. You call a plumber to ask about how much it will cost. Of course, the plumber cannot give you an exact cost but does give you the estimates shown in Table 6.1.

Since we know that the cost must depend on the time required, **cost** is the dependent variable and **time required** is the independent variable.

To decide if this situation can be described by a linear model, we need to see if the rate of change is constant. To do this, check the gradient (in this context, the cost per hour) between two or more pairs of points:

$$\frac{185 - 110}{2 - 1} = 75 \quad \text{and} \quad \frac{260 - 185}{3 - 2} = 75$$

Since the cost per hour is constant, a linear model is appropriate for this situation. In addition, we have discovered that the cost per hour is €75. However, there seems to be another part to the cost. We can use slope-intercept form to find a linear model. We will use  $y$  for the dependent variable, cost, and  $x$  for the independent variable, time:

$$y - y_1 = m(x - x_1) \Rightarrow y - 110 = 75(x - 1) \Rightarrow y = 75x + 35$$

We can see that the plumber's hourly rate is €75, and he adds a fixed amount of €35. This is probably to compensate him for travelling to your home! Finally, it's a good idea to test the model to make sure it describes the situation:

$$\text{For 1 hour: } y = 75(1) + 35 = 110$$

$$\text{For 2 hours: } y = 75(2) + 35 = 185$$

This matches the estimates given so we can be confident that our model is appropriate. Now we can see what we might have to pay if it takes the plumber a whole 8-hour day to fix our problem:

$$\text{For 8 hours: } y = 75(8) + 35 = \text{€}635$$

Notice also that graphs of linear functions are always lines.



It is important to remember that all models are simplifications of reality. We use models to tell us something about the way a system behaves and to make predictions. The goal of a model is to simplify and approximate a real system so that we can learn, predict, and analyse the behaviour. Part of our job is to use models wisely and be sure we understand the limitations and assumptions of any model we use.

Time required (hours)	Cost of repair (€)
1	110
2	185
3	260

Table 6.1 Repair cost estimates

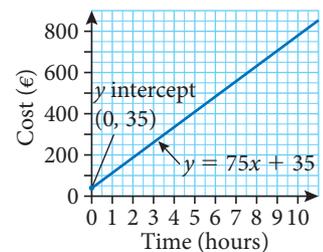


Figure 6.2 The graph of a linear model of the cost of hiring a particular plumber to come to your home

Remember that in a graph the independent variable is usually placed on the horizontal axis, and the dependent variable is placed on the vertical axis.



Linear models describe situations where the rate of change (gradient) is constant. The graph of a linear function is a line.

## Extending and revising models



**Figure 6.3** New York City tax rates decal from the late 1970s. Linear models are very well suited to this sort of situation

Sometimes we can make an initial simple model but need to revise it to be more useful. Here is an example.

Consider the taxi costs shown in Figure 6.3. Because the cost increases at a fixed rate relative to the distance driven (10 cents for each  $\frac{1}{7}$  miles driven), this is a good candidate for a linear model. Since the cost depends on the number of miles driven, we will make the cost the dependent variable and the distance driven the independent variable. It often helps to make a table showing the independent and dependent variables:

Miles driven	Calculation	Cost (\$)
$\frac{1}{7}$	$0.75 + 0$	0.75
$\frac{2}{7}$	$0.75 + 0.10(1)$	0.85
$\frac{3}{7}$	$0.75 + 0.10(2)$	0.95
$\frac{4}{7}$	$0.75 + 0.10(3)$	1.05

**Table 6.2** Independent and dependent variables

By explicitly showing our calculations, we get a good idea of how to develop our model. From the table, it appears that an appropriate model is  $\text{Cost} = 0.75 + 0.10x$ . But be careful! In this case, what is  $x$ ? Notice that the number we are multiplying by 0.10 is not the number of miles driven – it is the number of  $\frac{1}{7}$  miles after the first  $\frac{1}{7}$  mile!

Since most people don't think in terms of  $\frac{1}{7}$  miles, it would be useful if our independent variable was simply distance in miles. Let's try to revise our table:

Miles driven	The number of $\frac{1}{7}$ miles after the first $\frac{1}{7}$ mile	Calculation	Cost (\$)
$\frac{1}{7}$	$\frac{1}{7} - \frac{1}{7} = 0$	$0.75 + 0.10(0)$	0.75
$\frac{2}{7}$	$7\left(\frac{2}{7} - \frac{1}{7}\right) = 1$	$0.75 + 0.10(1)$	0.85
$\frac{3}{7}$	$7\left(\frac{3}{7} - \frac{1}{7}\right) = 2$	$0.75 + 0.10(2)$	0.95
$\frac{4}{7}$	$7\left(\frac{4}{7} - \frac{1}{7}\right) = 3$	$0.75 + 0.10(3)$	1.05
$m$	$7\left(m - \frac{1}{7}\right)$	$0.75 + 0.10x$	<b>C</b>

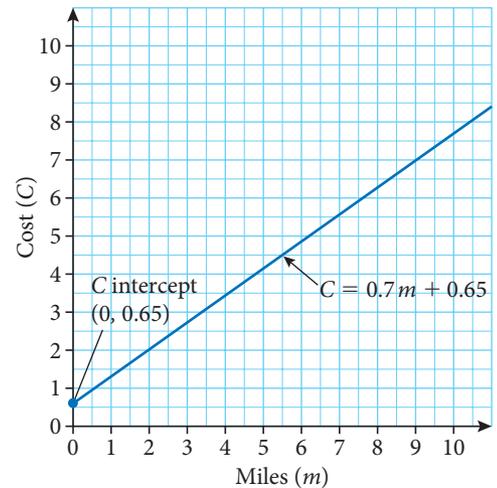
**Table 6.3** Revised table

To change 'miles driven' into 'the number of  $\frac{1}{7}$  miles after the first  $\frac{1}{7}$  mile,' we need to subtract  $\frac{1}{7}$  (representing the first  $\frac{1}{7}$  mile) and then multiply by 7 (so that each  $\frac{1}{7}$  mile is counted as one unit). So now we know that  $C = 0.75 + 0.10x$  and  $x = 7\left(m - \frac{1}{7}\right)$  where  $m$  is the number of miles.

By substituting the equation for  $x$  into the equation for  $C$ , we get

$$C = 0.75 + 0.10\left(7\left(m - \frac{1}{7}\right)\right) \Rightarrow C = 0.7m + 0.65$$

A graph of this function is shown in Figure 6.4.



**Figure 6.4** A graph of the linear model for the cost of hiring a New York taxi in the 1970s

## Models don't always capture reality perfectly

If we look at the graph of our model from the taxi cost example, we see that the model suggests that for a journey of 0 miles, we will pay \$0.65. However, we know that we will always pay at least \$0.75. What went wrong?

The problem here is that the model assumes that the incremental cost (represented by the gradient of the function) is continuous – that is, that the taxi will charge us for any increment of a mile. However, we know that the taxi will charge for each  $\frac{1}{7}$  of a mile. To make this point clear, consider what happens if we drive 0.5 miles. The model suggests that the cost would be

$$C = 0.7(0.5) + 0.65 = \$1.00$$

But we know that 0.5 miles is more than  $\frac{3}{7}$  of a mile and less than  $\frac{4}{7}$  so we would actually get charged

$$C = 0.7\left(\frac{4}{7}\right) + 0.65 = \$1.05$$

That is, our model works as long as we round the miles up to the nearest  $\frac{1}{7}$  of a mile. As stated in the introduction, all models are a simplification of reality. This model can help us see how the cost of the ride relates to the length of the ride, but we need to be careful in using it to predict exact costs.

### Example 6.1

The I&T Fitness Centre charges a one-time joining fee of \$100 and then charges \$2 per visit.

- Develop a model for the total cost after  $v$  visits.
- Use your model to find the cost of 20 visits.
- Draw a graph of your model.

We have drawn the graph with a continuous line, even though it is only possible to visit a whole number of times – e.g., we can't visit 1.5 times.

For convenience, we often draw the graph as a continuous line even though that may not strictly represent reality.

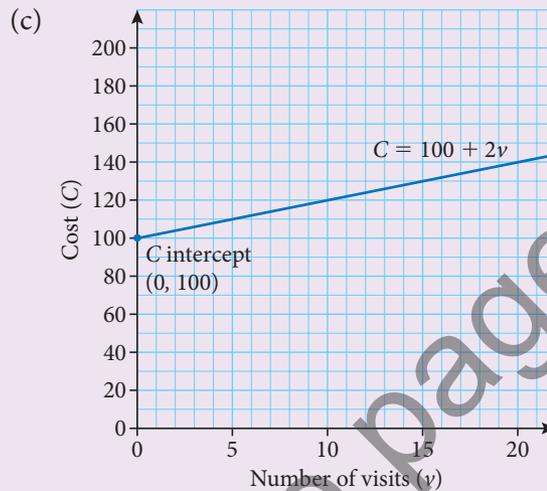
### Solution

- (a) Since the rate is already given (\$2 per visit), plus a one-time cost of \$100, we can write the model directly:

$$C = 100 + 2v$$

where  $C$  is the total cost and  $v$  is the number of visits.

- (b) Using our model, 20 visits would cost  $C = 100 + 2(20) = \$140$



### Example 6.2

A visitor to the I&T Fitness Centre decides not to buy a membership and instead just pays the daily rate. The first visit costs her \$12. At the end of the month, she notices she has visited 5 more times and paid a total of \$60 for those 5 visits.

- (a) Develop a linear model for the total cost after  $v$  visits.  
 (b) Use your model to predict the cost of 20 visits.  
 (c) After how many visits is it better to buy the membership described in Example 6.1?

### Solution

- (a) One visit costs \$12, and 5 visits cost \$60. To be sure there are no extra fees, check that the cost per visit for 5 visits is the same as the cost for 1 visit. For 5 visits, the cost per visit is  $\frac{60}{5} = \$12$

Therefore, the rate of change is constant, and we can develop the model:

$C = 12v$  where  $C$  is the total cost and  $v$  is the number of visits.

- (b) Using our model, 20 visits would cost  $C = 12(20) = \$240$

This is much more expensive than the membership plan in Example 6.1!

- (c) To find out at which point the membership plan in Example 6.1 becomes the better option, we first find at which point the two plans are equal. We can do this algebraically or graphically.

Algebraically, we are looking for the number of visits ( $v$ ) that produces the same cost. Therefore, we can write  $C = 100 + 2v = 12v$  and solve for  $v$ :

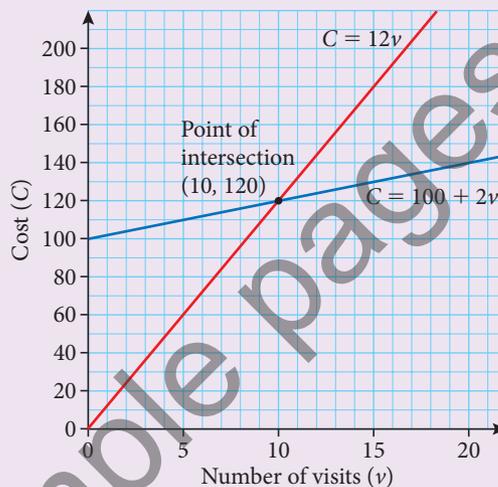
$$100 + 2v = 12v \Rightarrow v = 10$$

Therefore, the two plans are the same for 10 visits. Since we know that the membership plan costs only \$2 per visit, we know that it will be cheaper for any number of visits more than 10.

We can graph both models and look for the intersection.

The intersection point of the two graphs tells us that the two plans will cost the same (\$120) at 10 visits.

We can see clearly that the membership plan increases much more slowly: \$2 per visit instead of \$12 per visit, so it is cheaper after 10 visits.



## Interpreting and evaluating linear models

It is important to be able to recognise the structure of a linear model and interpret its meaning. Also, we must be careful to recognise limitations of linear models.

### Example 6.3

The number of items that a clothing store sells can be modelled by the function  $N = 1000 - 5p$ , where  $N$  is the number of items sold and  $p$  is the price of the jeans in euros.

- Use the model to predict the number of jeans sold when the jeans are priced at €100.
- Interpret the value of the gradient and  $N$ -intercept in context.
- Interpret the value of the  $p$ -intercept in context.
- Use the model to predict the number of jeans sold when the jeans are priced at €500. Give a reason why this prediction is not reasonable.

Example 6.3 shows that linear models often become nonsensical for certain extreme values of the independent variable.

For this reason, it's best to give a limitation on the domain of the model to avoid these silly results.

In the model above, it is sensible to limit the domain of the model to  $1 \leq p \leq 200$

The upper bound of  $p = 200$  is when the model predicts that  $N$  will be zero.

### Solution

- (a) The model predicts that the number of jeans sold when the jeans are priced at 100 euros is:  $N = 1000 - 5(100) = 500$  jeans.
- (b) The gradient of  $-5$  represents that for each euro that the price increases, the number of jeans sold decreases by 5 jeans. The  $N$ -intercept of  $(0, 1000)$  theoretically represents that the number of jeans sold when the jeans are free will be 1000 jeans – probably not realistic!
- (c) The  $p$ -intercept occurs when  $N = 0$ . Therefore, we must solve  $0 = 1000 - 5p$  to obtain  $1000 = 5p \Rightarrow p = 200$   
This tells us that when the price of jeans is 200 euros, the number of jeans sold will be 0.
- (d) The model predicts that the number of jeans sold when the jeans are priced at €500 will be:  $N = 1000 - 5(500) = -1500$  jeans  
This is not reasonable because this suggests that customers will be giving back their jeans!

## Piecewise linear models

Sometimes a real-life situation is not modelled by a single linear function, but is linear in parts. Consider the following example:

### Example 6.4

A phone company charges a rate of \$0.24 for the first minute of a call, then \$0.12 per minute for the next 9 minutes, then \$0.06 per minute thereafter. Calls are charged on a per-second basis.

- (a) Develop a piecewise linear model for the cost  $C$  of a call lasting  $t$  seconds.  
 (b) Use your model to calculate the cost of calls lasting:  
 (i) 45 seconds      (ii) 4 minutes      (iii) 15 minutes.

### Solution

- (a) For  $0 < t \leq 60$ , the cost is \$0.24 per minute, or  $\frac{0.24}{60} = \$0.004$  per second. Then, for  $60 < t \leq 600$ , the cost is \$0.12 per minute or  $\frac{0.12}{60} = \$0.002$  per second. However, we must also add in the cost for the first minute, and not charge twice for the first minute, so in total the cost will be  $0.24 + 0.002(t - 60)$  for a call of  $t$  seconds. Likewise, for calls more than 10 minutes, we have the cost of the first minute, plus the cost of the next 9 minutes, plus the remaining cost of  $\frac{0.06}{60} = 0.001$  per minute, which gives us  $0.24 + 0.12(9) + 0.001(t - 600)$ , which simplifies to  $1.32 + 0.001(t - 600)$  for a call of  $t$  seconds.

We express this mathematically with the following notation:

$$C = \begin{cases} 0.004t, & 0 < t \leq 60 \\ 0.24 + 0.002(t - 60), & 60 < t \leq 600 \\ 1.32 + 0.001(t - 600), & t > 600 \end{cases}$$

In this notation, we define the value of  $C$  through a **piecewise function**.

To evaluate the function, we simply select the piece of the function that applies to the value of  $t$  we want, as in part (b).

- (b) For each value of  $t$ , we simply select the appropriate piece of the function.
- Since  $t = 45$  is between 0 and 60 seconds, we use  $C = 0.004t$  to give us  $C = 0.004(45) = \$0.18$
  - Since 4 minutes is 240 seconds, and  $t = 240$  is between 60 and 600 seconds, we use  $C = 0.24 + 0.002(t - 60)$  to give us  $C = 0.24 + 0.002(240 - 60) = \$0.60$
  - Since 15 minutes is  $15 \times 60 = 900$  seconds, and  $t = 900$  is more than 600 seconds, we use  $C = 1.32 + 0.001(t - 600)$  to give us  $C = 1.32 + 0.001(900 - 600) = \$1.62$

Many calculators can graph piecewise functions, as shown in Figure 6.5.

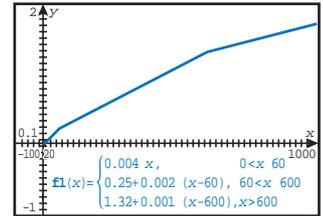


Figure 6.5 GDC piecewise function

### Exercise 6.1

- A plane is currently 3000 km from its destination, travelling at a constant speed of  $900 \text{ km h}^{-1}$ .
  - Develop a linear model for the distance  $d$  remaining after  $t$  hours of travel.
  - Interpret the  $d$ -intercept of your model in context.
  - Interpret the  $t$ -intercept of your model in context.
  - State a reasonable domain and range for your model.
- A plane is currently 5000 km from its destination. 1.5 hours later, it is 3800 km from its destination.
  - Develop a linear model for the distance  $d$  remaining after  $t$  hours of travel.
  - Interpret the gradient of your model in context.
  - Interpret the  $d$ -intercept of your model in context.
  - Interpret the  $t$ -intercept of your model in context.
  - State a reasonable domain and range for your model.
- The table shows a comparison between EU and USA shoe sizes.

USA (Men's)	7	8	9
EU	40	41	42

- Develop a linear model to find the EU shoe size given the USA shoe size.
- Use your model to predict the EU shoe size for a USA Men's shoe size of 12.

- (c) Use your model to predict the USA Men's shoe size for an EU shoe size of 44.
- (d) Interpret the gradient of your model in context.
- (e) Given that USA Men's shoe sizes typically run from 6 to 16, calculate a reasonable domain and range for your model.

4. Julie has collected data on how long it takes her to read books, based on the number of pages. The data she collected are shown in the table.

<b>Number of pages</b>	340	290	500
<b>Time to read (minutes)</b>	490	420	714

- (a) Develop a linear model for the time required to read  $n$  pages.
  - (b) Use your model to predict the time required to read 1000 pages. Give your answer to the nearest 10 minutes.
  - (c) Interpret the gradient and  $y$ -intercept of your model in context.
  - (d) State a reasonable domain and range for your model.
5. Given that  $68^\circ\text{F} = 20^\circ\text{C}$  and  $212^\circ\text{F} = 100^\circ\text{C}$ :
- (a) Develop a linear model for degrees Fahrenheit ( $F$ ) in terms of degrees Celsius ( $C$ ).
  - (b) Explain, in context, what the gradient of your model represents.
  - (c) Interpret the  $F$ -intercept of your model in context.
  - (d) Interpret the  $C$ -intercept of your model in context.
  - (e) Use your model to convert  $10^\circ\text{C}$  into  $^\circ\text{F}$ .
  - (f) Use your model to find the numerical value in  $^\circ\text{C}$  that is the same in  $^\circ\text{F}$ .
  - (g) Given that absolute zero (the lowest possible temperature) is  $-273^\circ\text{C}$ , calculate a reasonable domain and range for your model.
6. The DJ, IB Cool, charges a flat fee of \$150 per party plus \$75 per hour. The DJ, MC Numbers, charges \$120 per party plus \$80 per hour.
- (a) Find linear models for each DJ as a function of the length of the party in hours.
  - (b) For parties longer than  $n$  hours, IB Cool is less expensive. Find the value of  $n$ .
7. A cyclist pedals at the rate of  $300 \text{ m min}^{-1}$  for 20 minutes, then slows down to  $150 \text{ m min}^{-1}$  for 16 minutes, then races at  $400 \text{ m min}^{-1}$  for 4 minutes.
- (a) Find the distance travelled after:
    - (i) 20 minutes
    - (ii) 36 minutes
    - (iii) 40 minutes.
  - (b) Write a piecewise linear function for the distance  $D(t)$  in terms of the time  $t$  in minutes.
  - (c) Find the distance travelled after:
    - (i) 30 minutes
    - (ii) 38 minutes.
  - (d) When has the cyclist travelled:
    - (i) 8 km
    - (ii) 9 km?

8. The Athabasca Glacier in Alberta, Canada, has been slowly shrinking for many years. Photos from 1844 show the glacier about 2 km longer than it was in 2018.

- (a) Calculate the average rate at which the glacier is shrinking.
- (b) In 1844, the glacier was 8 km long. Assuming the present rate continues, in what year will the glacier be gone?

In the year 1900, the end of the glacier was about 300 km from the US–Canada border; as it shrinks it moves farther from the border.

- (c) Write a function for the distance  $d$  in km between the end of the glacier and the US–Canada border, in terms of time  $t$ , in years, since 1900.
- (d) Assuming that the rate of change has been constant for many years, at what time did the glacier reach the US–Canada border?

9. Two plastic cup factories, Cups R Us and Cupomatic, can produce cups printed with the image of your choice. At Cups R Us, the mandatory setup and design cost is ZAR350 and the cost per cup is ZAR8.50.

- (a) Develop a linear model for the cost,  $C$ , of an order at Cups R Us based on the number of cups,  $n$ .
- (b) Write down a reasonable domain and range for your model.
- (c) Use your model to calculate the cost of an order of:
  - (i) 100 cups
  - (ii) 200 cups
  - (iii) 400 cups.
- (d) Calculate the average cost per cup for:
  - (i) 100 cups
  - (ii) 200 cups
  - (iii) 400 cups.
- (e) Hence, give a reason why, in general, it is more cost-effective to order more cups.

Cupomatic charges ZAR2150 for 200 cups and ZAR3750 for 400 cups.

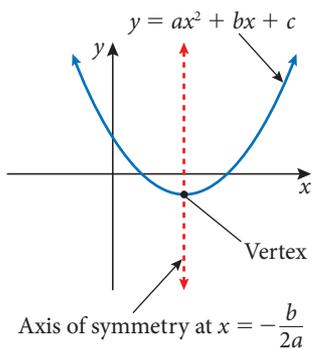
- (f) Develop a linear model for the cost,  $D$ , of an order at Cupomatic based on the number of cups,  $n$ .
- (g) Write down a reasonable domain and range for your model.
- (h) Interpret the gradient of your model in context.
- (i) Use your linear model to predict the cost of 600 cups.
- (j) For orders of more than  $x$  cups, it is more cost-effective to order from Cupomatic. Find the value of  $x$ .

10. Continuing from the previous question, Cups R Us will waive the setup and design cost if the order is at least 500 cups.

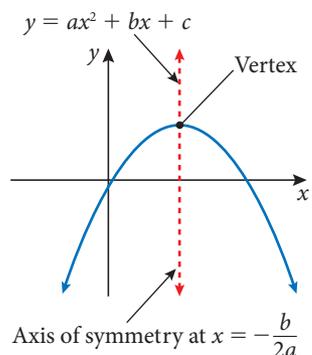
- (a) Develop a piecewise model for the cost,  $C$ , of an order at Cups R Us based on the number of cups,  $n$ .
- (b) The main competitor of Cups R Us is Cupomatic. You are given that the model for the cost,  $D$ , of ordering  $n$  cups from Cupomatic is  $D = 8n + 550$ . It is less expensive to order from Cupomatic if  $x$ , the number of cups ordered, is in the intervals  $a \leq x < b$  or  $x > k$ . Find the values of  $a$ ,  $b$ , and  $k$ .

11. As of 2018, taxi cab tariffs for working hours in London, England, are as follows.

- For the first 234.8 metres or 50.4 seconds (whichever is reached first) there is a minimum charge of 2.60 GBP.
  - For each additional 117.4 metres or 25.2 seconds (whichever is reached first), or part thereof, if the distance travelled is less than 9656.1 metres there is a charge of 0.20 GBP.
  - Once the distance has reached 9656.1 metres then there is a charge of 0.20 GBP for each additional 86.9 metres or 18.7 seconds (whichever is reached first), or part thereof.
- (a) Develop a piecewise linear model for the cost,  $C$ , of a taxi ride based on the distance travelled,  $m$ , in metres.
  - (b) Find the cost of 0.2 km, 5 km, and 15 km rides.
  - (c) Develop a piecewise linear model for the cost,  $D$ , of a taxi ride based on the time taken,  $t$ , in seconds, ignoring distance.
  - (d) Find the cost of 0.5 minute, 5 minute, and 15 minute rides.
  - (e) Given that the actual taxi fare is always the greater of the two models, find:
    - (i) the cost of a ride that takes 10 minutes to go 4 km
    - (ii) the cost of a ride that takes 5 minutes to go 4 km.



**Figure 6.6** If  $a > 0$  then the parabola is concave up



**Figure 6.7** If  $a < 0$  then the parabola is concave down

## 6.2 Quadratic models

Quadratic models appear frequently in real-world situations involving area, economics, projectile motion, and falling objects, among others. In this section we will look primarily at quadratic models of the form  $y = ax^2 + bx + c$

Quadratic models are used in situations where the rate of change in the dependent variable changes linearly with respect to the independent variable.

For example:

- Falling objects due to gravity, projectile motion: the acceleration is constant, the velocity follows a linear model, and the displacement (position) follows a quadratic model.
- Revenue models: the number of items sold of some item based on the price follows a linear model; the revenue from selling the number of items follows a quadratic model.

Quadratic models also have some geometric properties that make them well-suited to designing satellite dishes and to modelling bridge spans.

Before we begin, we will restate key properties of quadratic functions that you have seen before.



For a quadratic function of the form  $y = ax^2 + bx + c$ , where  $a \neq 0$

- The graph of a quadratic function is roughly U-shaped and is called a **parabola**.
- Concavity: The graph of the function is **concave up** if and only if  $a > 0$   
It is **concave down** if and only if  $a < 0$
- Symmetry: The graph of the function is symmetrical about the vertical line with equation  $x = -\frac{b}{2a}$   
This line is called the **axis of symmetry**.
- Maximum/minimum: The function has a **maximum** (when concave down) or **minimum** (when concave up) where the graph intersects the axis of symmetry. This point is called the **vertex** of the parabola. The  $x$  coordinate of the vertex is therefore given by  $x = -\frac{b}{2a}$
- The  **$y$ -intercept** of the graph is found at  $(0, c)$
- The  **$x$ -intercepts** of the graph, also called the **zeros** of the function, can be found with the quadratic formula:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$   
Remember that a graph may have zero, one, or two  $x$ -intercepts.

### Example 6.5

A ball is thrown upwards from the top of a building. The height of the ball from ground level can be modelled by the function  $h(t) = -4.9t^2 + 11t + 50$  where  $h(t)$  is the height of the ball in metres and  $t$  is the time in seconds after the ball was thrown.

- Sketch a graph of the function.
- Write down the height of the building.
- Find the time when the ball reaches its maximum height.
- Find the maximum height reached by the ball.
- Find the time when the ball hits the ground.
- Describe a reasonable domain and range for this model.

### Solution

We can solve this problem either by using an algebraic approach ('by hand') or by using our GDC to analyse the graph.

#### Algebraic approach

- We know the ball travels upwards before coming back down and we know the parabola is concave down since  $a < 0$
- The height of the building is the initial height of the object, which is the height of the ball when  $t = 0$ , in other words, the  $y$  intercept. Therefore, the height of the building is 50 m.
- The time when the ball reaches its maximum height is the  $t$ -coordinate of the vertex of the parabola. We start by finding the  $t$  coordinate, which is also the location of the axis of symmetry:

$$t = -\frac{b}{2a} = -\frac{11}{2(-4.9)} = 1.12 \text{ s (3 s.f.)}$$

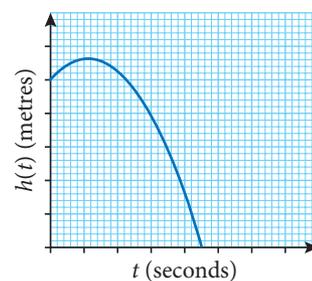


Figure 6.8 Solution to Example 6.5 (a)

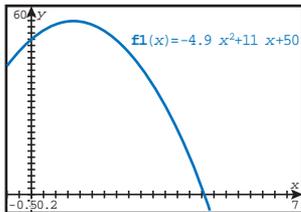


Figure 6.9 GDC approach (a)

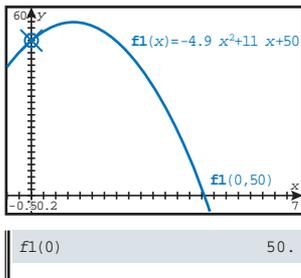


Figure 6.10 GDC approach (b)

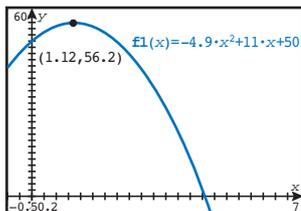


Figure 6.11 GDC approach (c)

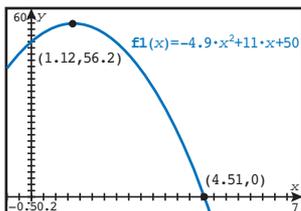


Figure 6.12 GDC approach (e)

- (d) The maximum height reached by the ball is found by substituting the  $t$  value from (c) back into the function:

$$h(1.12) = -4.9(1.12)^2 + 11(1.12) + 50 = 56.2 \text{ m (3 s.f.)}$$

- (e) The ball hits the ground when height is zero. Therefore, we must solve the equation  $0 = -4.9t^2 + 11t + 50$

We can use the quadratic formula to solve this:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-11 \pm \sqrt{11^2 - 4(-4.9)(50)}}{2(-4.9)} = 4.51 \text{ s (3 s.f.)}$$

- (f) We should limit the domain to non-negative values. Also, the model doesn't make sense after the ball hits the ground since the graph suggests that the ball goes underground. Therefore a reasonable domain for this model is  $0 \leq t \leq 4.51$

The range indicates the possible heights of the ball. Here,  $0 \leq h(t) \leq 56.2$  makes sense since we know the maximum height of the ball and we presume the ball does not go underground.

### GDC approach

- (a) Here we use the GDC to obtain a graph, as shown in Figure 6.9.

We need to be careful to adjust the viewing window appropriately.

On our calculator, we need to use  $x$  in place of  $t$ . It is important to note that this graph is height versus time – it is not the trajectory (path) of the object. The horizontal axis represents time, not distance.

- (b) The Graph Trace feature can be used to evaluate the function at  $t = 0$  to find the initial height, as shown in Figure 6.10. We could also evaluate the function directly.

- (c) We can use our GDC to find the maximum height, as in Figure 6.11.

We see that the  $x$  coordinate is 1.12, so the ball reaches a maximum height at 1.12 s.

- (d) Using the same point from part (c), we see that the  $y$  coordinate is 56.2, so the ball's maximum height is 56.2 m.

- (e) We can use our GDC to find the positive zero of the function, as in Figure 6.12. The ball hits the ground after 4.51 s.

- (f) We use the same logic as in the algebraic approach to conclude that a reasonable domain is  $0 \leq t \leq 4.51$  and a reasonable range is  $0 \leq h(t) \leq 56.2$

In the next few examples we will look at building a quadratic function.

### Example 6.6

The number of jeans sold by a clothing store can be modelled by the function  $N = 1000 - 5p$ , where  $N$  is the number of items sold and  $p$  is price of the jeans in euros.

- Develop a model for the revenue earned from selling these jeans based on the selling price  $p$ .
- Use your model to find the price the jeans should be sold at in order to maximise revenue.
- Find the maximum revenue predicted by your model.
- Give a reasonable domain and range for your model.

### Solution

- (a) In general, revenue = selling price  $\times$  number of units sold. Therefore, we can develop a model for revenue by multiplying the selling price,  $p$ , by the expression for the number of jeans sold:

$$R = (p)(1000 - 5p) \Rightarrow R = -5p^2 + 1000p$$

- (b) Since this is a quadratic model with a concave-down graph ( $a < 0$ ), we know there will be a maximum at the vertex. In this case, the  $p$  coordinate of the vertex is the selling price and the  $R$  coordinate is the revenue. To find the price, we use the formula for the  $p$  coordinate of the vertex:

$$p = -\frac{b}{2a} = -\frac{1000}{2(-5)} = 100 \text{ euros}$$

- (c) To find the maximum revenue, we need to evaluate the model for 100 jeans (from part (b)):

$$R = -5(100)^2 + 1000(100) = 50\,000 \text{ euros}$$

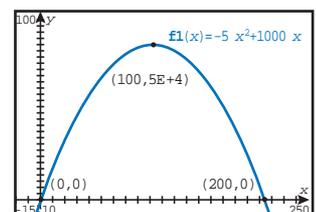
- (d) Clearly, our model doesn't make much sense if the price  $p$  is less than zero. But what about a maximum price? Since we know that this is a concave-down quadratic function with a vertex above the  $p$  axis, we know there will be two  $p$  intercepts. Solve for them algebraically:

$$0 = -5p^2 + 1000p \Rightarrow 0 = (-5p)(p - 200) \Rightarrow p = 0, p = 200$$

Therefore the  $p$  intercepts are 0 and 200. So  $0 \leq p \leq 200$  is a reasonable domain. The range is given by the minimum and maximum values of the function on this domain:  $0 \leq R \leq 50\,000$

It's often helpful to generate a graph of a model to get a picture of it. In this case, we can clearly see the maximum revenue and can use our GDC to verify our results in (c) and (d), as shown in Figure 6.13.

Note that  $5E+4$  for the  $y$  coordinate of the vertex is the GDC's version of scientific notation. We should read it like this:  $5E+4 = 5 \times 10^4 = 50\,000$



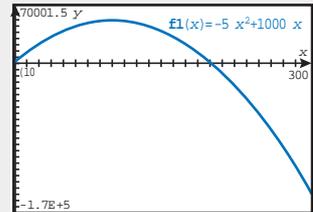
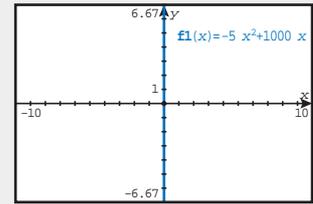
**Figure 6.13** The graph of the model for revenue vs price of jeans

When using a GDC to analyse a model graphically, it's important to take care when setting the viewing window. The default view for many GDCs is  $-10 \leq x \leq 10$  and  $-10 \leq y \leq 10$  or smaller. This view can be very misleading. For example, the model in Example 6.5 looks like this with the default viewing window.

The graph is there but with the current settings it is almost indistinguishable from the  $y$  axis. We can use our knowledge of the function to choose more appropriate settings.

After setting the  $x$ -axis to a suitable domain, in this case  $0 \leq x \leq 300$ , and using the Zoom Fit feature to scale the  $y$ -axis to fit the function, we get the image shown.

The negative part of the graph isn't very useful, but you can then use Zoom Box to examine the function more accurately.

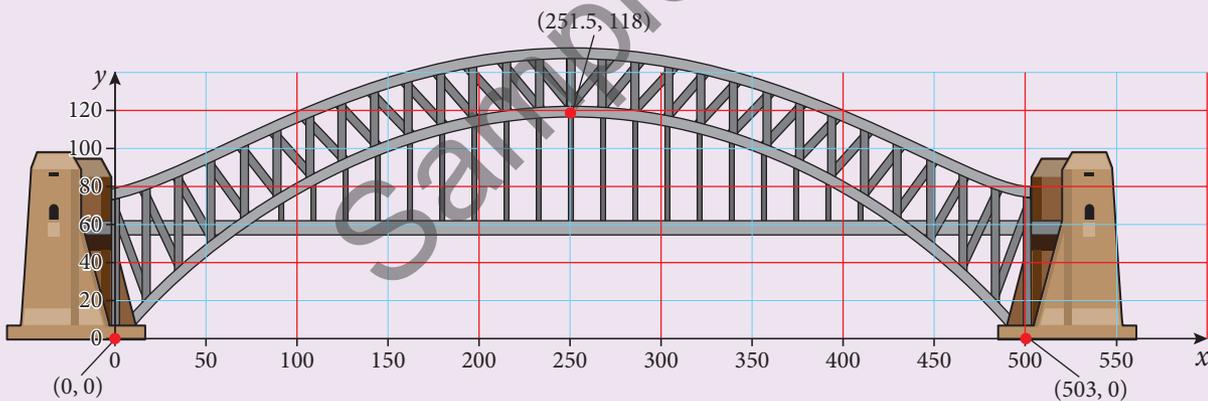


### Example 6.7

The Sydney Harbour Bridge is supported by two spans that can be modelled by quadratic functions. The lower span is approximately 503 m wide and 118 m tall at its highest point. Develop a quadratic model for the lower span such that one end of the span is positioned at  $(0, 0)$ .

### Solution

We can start by drawing our axes and locating the vertex at the maximum point of the lower span. Since the vertex must be halfway along the length of the bridge, its  $x$  coordinate must be 251.5.



Then we can use the general quadratic model  $y = ax^2 + bx + c$  and some algebra to find the model for the lower span. First, since the graph must pass through  $(0, 0)$ , it must be true that

$$0 = a(0)^2 + b(0) + c \Rightarrow c = 0$$

Upon reflection, we could have deduced the value of  $c$  from recognising that the  $y$  intercept of the graph is at  $(0,0)$ . So far, our model is therefore

$$y = ax^2 + bx$$

Next, since the graph must also pass through  $(251.5, 118)$  and  $(503, 0)$ , we can generate a system of equations by substituting each coordinate pair into the model:

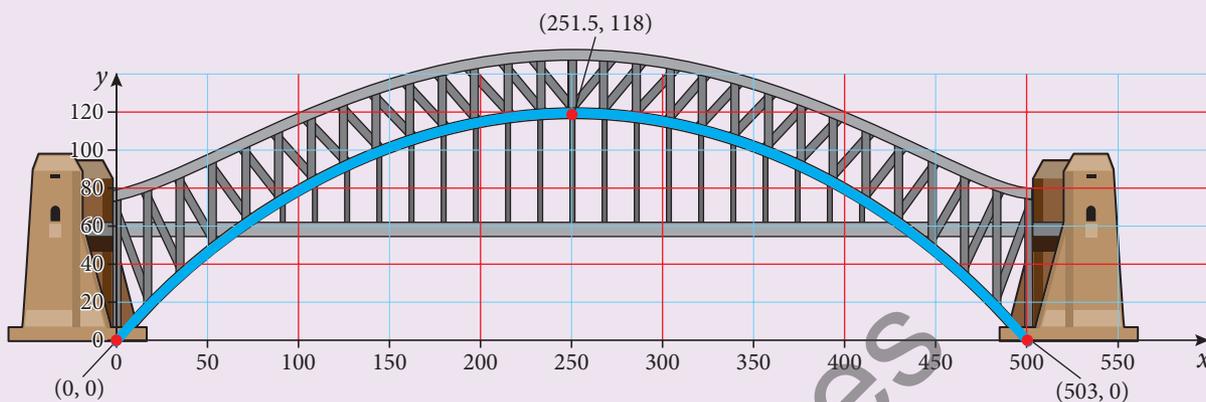
$$\begin{cases} 118 = a(251.5)^2 + b(251.5) \\ 0 = a(503)^2 + b(503) \end{cases} \Rightarrow \begin{cases} 63\,252.25a + 251.5b = 118 \\ 25\,3009a + 503b = 0 \end{cases}$$

We can solve this system (algebraically, or by using a GDC) to obtain:

$$a = -0.001866, b = 0.9384$$

Therefore, our model is  $y = -0.001866x^2 + 0.9384x$

If we graph this function with appropriate axes, we can see that it fits the lower span very well:

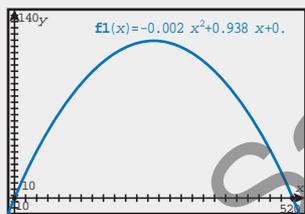


Note that most GDCs have a **quadratic regression** feature that can also find the model.

Notice that the values for  $a$  and  $b$  calculated by the GDC agree with our values.

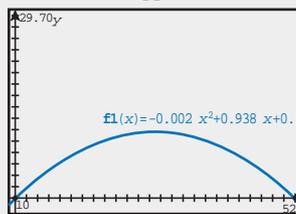
	x	y		
1				=QuadReg(
2	0	0	Title	Quadratic...
3	251.5	118	RegEqn	a*x^2+b*x...
4	503	0	a	-0.001866
5			b	0.93837
6			c	0.

Remember that the graph may appear differently depending on the viewing window chosen on your graph. For example, in the first screenshot, the scale of the units on the  $x$  and  $y$  axes is not 1:1, so the function appears taller than it should.



A graph can appear distorted if the  $x$  and  $y$  axes are not scaled in a 1:1 ratio

Using the GDC's Zoom Square function fixes this, as shown in the second screenshot.



Using the Zoom Square feature changes the  $x$  and  $y$  axes to a 1:1 ratio

## Exercise 6.2

1. On Earth, the position of a falling object can be modelled by the function  $h(t) = -4.9t^2 + v_0t + h_0$  where  $h(t)$  is the height in metres after  $t$  seconds,  $v_0$  is the initial velocity and  $h_0$  is the initial height.
  - (a) Write a model for the height of a ball thrown upwards with an initial velocity of  $5 \text{ m s}^{-1}$  from the roof of a 60-metre tall building.
  - (b) Use your model to find:
    - (i) the maximum height of the ball
    - (ii) the time until the ball hits the ground
    - (iii) the interval of time for which the ball is more than 50 metres above the ground.

Remember that  
profit = revenue - cost.

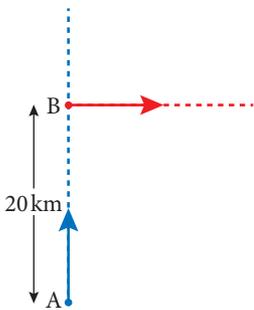


Figure 6.14 Diagram for question 3

2. A small manufacturing company makes and sells  $x$  machines each month. The monthly cost,  $C$ , in dollars, of making  $x$  machines is given by

$$C(x) = 0.35x^2 + 3200$$

The monthly revenue,  $R$ , in dollars, obtained by selling  $x$  machines is given by  $R(x) = 180x - 0.55x^2$

- Show that the company's monthly profit can be calculated using the quadratic function  $P(x) = -0.9x^2 + 180x - 3200$
- The maximum profit occurs at the vertex of the function  $P(x)$ . How many machines should be made and sold each month for a maximum profit?
- If the company does maximise profit, what is the selling price of each machine?
- Find the smallest number of machines the company must make and sell each month in order to make a positive profit.

3. The diagram shows two ships, A and B. At noon, ship A was 20 km due south of ship B. Ship A was moving north at  $10 \text{ km h}^{-1}$  and ship B was moving east at  $4 \text{ km h}^{-1}$

Find the distance between the ships at:

- 13:00
- 14:00

Let  $s(t)$  be the distance between the ships  $t$  hours after noon, for  $0 \leq t \leq 4$

- Show that  $s(t) = \sqrt{116t^2 - 400t + 400}$

- Sketch the graph of  $s(t)$

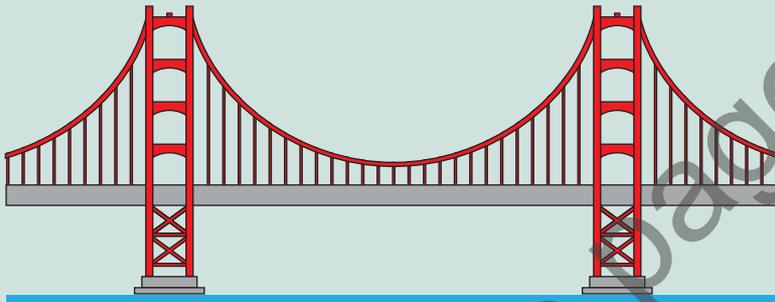
- Due to poor weather, the captain of ship A can see another ship only if they are less than 9 km apart.

- Find the values of  $t$  during which ship A can see ship B.
- Write down the times between which ship A can see ship B.

4. Worldwide grain production for the years 1965 to 2000 can be modelled by the function  $G = -0.144t^2 + 6.88t + 266$ , where  $G$  is the number of kilograms per person and  $t$  is years since 1965.

- Based on this model, what was the grain production in 1975?
- What was the maximum level of grain production and when did it occur?
- The actual worldwide grain production in 2005 was 10 kg per person. What does this model predict for 2005?
- If worldwide grain production drops below 100 kg per person, major economic and health consequences are possible. Based on this model, when might this occur?
- Give a reason why this model will not (we hope) continue to be true.

5. The density of water based on temperature follows a quadratic model. The maximum density of water is  $1 \text{ g mm}^{-3}$  at  $4^\circ\text{C}$ . At  $80^\circ\text{C}$ , the density is  $0.97183 \text{ g mm}^{-3}$ .
- Find a quadratic model for density,  $D$ , in terms of temperature,  $T$ , in the form  $D = a(T - h)^2 + k$  where  $a$ ,  $h$ , and  $k$  are constants to be determined.
  - Use your model to find the density of water at  $0^\circ\text{C}$  to 5 significant figures.
  - At what temperature  $t > 0$  does the density of water drop below  $0.960 \text{ g mm}^{-3}$ ?
6. The main span of the Verrazano-Narrows bridge in New York City has a central span that is about 1300 m wide.



The main cable supporting this span is about 150 m above the roadway at the top of each suspension tower, and about 6 m above the roadway at its lowest point in the centre.

- Find a quadratic function to model the height  $h$  of the cable above the roadway in terms of the distance  $d$  from the left suspension tower.
  - Write down the domain and range of your function.
  - Calculate the height of the cable at a point 100 m from the left suspension tower.
  - At what distances from the left suspension tower is the cable less than 50 m above the roadway?
7. A farmer wants to fence two identical adjacent fields as shown in Figure 6.15. He has 1200 m of fencing to enclose the two identical regions.
- Write down an expression for the total area,  $A$ , in terms of  $x$ .
  - Find the maximum total area for the two fields and the dimensions  $x$  and  $y$ .
8. A shop sells t-shirts for €16 each and sells 40 t-shirts per day. Analysing past sales shows that for every euro increase in price, the shop sells 2 fewer t-shirts per day. Let  $x$  be the price increase in euros.
- Write an expression for the price of a single t-shirt in terms of  $x$ .
  - Write an expression for the number of t-shirts the shop can expect to sell per day in terms of  $x$ .

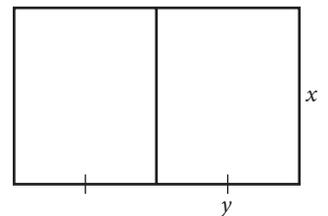


Figure 6.15 Diagram for question 7